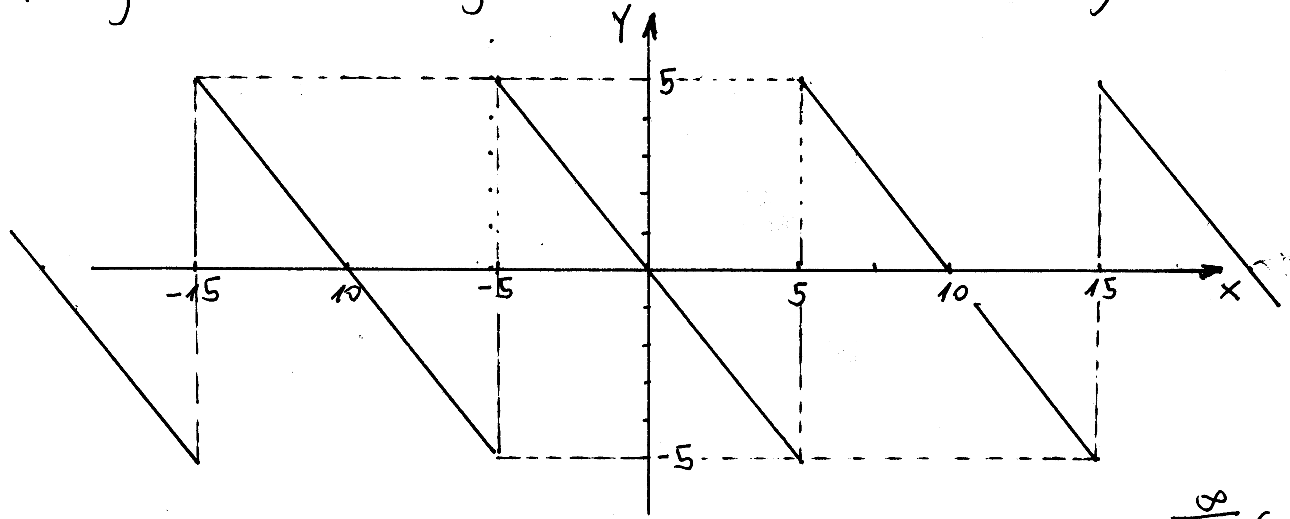
Funkciju definisanu grafikom pretvoriti u Furijeov red.



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi}{50}$.

Rj: Primjetimo da je data f-ja periodična perioda 10. Prema tome dovoljno ju je pretvoriti u Furijeov red na proizvoljnom intervalu periodu 10. Pa posmatrajmo npr. interval $[-5,5]$. F-ja na ovom intervalu ima oblik $f(x)=-x$.

Furijeov red f-je $f(x)$ na intervalu $[a,b]$ ima oblik:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right)$$

gdje su $a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$, $a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx$ i $b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$

$n=1,2,\ldots$

Furijeovi koeficijenti. U našem slučaju interval $[a,b]$ je $[-5\,5]$ pa je $b-a=5+5=10$, $\frac{2}{10}=\frac{1}{5}$, $\frac{2n\pi x}{b-a} = \frac{2n\pi x}{10} = \frac{n\pi x}{5}$.

$$a_0 = \frac{1}{5}\int_{-5}^{5}(-x)\,dx = \frac{1}{5}(-1)\cdot\frac{1}{2}x^2\Big|_{-5}^{5} = 0$$

$d\left(\frac{n\pi x}{5}\right) = \frac{n\pi}{5}dx$

$$a_n = \frac{1}{5}\int_{-5}^{5}(-x)\cos\frac{n\pi x}{5}\,dx = \begin{vmatrix} u=x & dv=\cos\frac{n\pi x}{5}dx \\ du=dx & v=\frac{5}{n\pi}\sin\frac{n\pi x}{5}\end{vmatrix} =$$

$$= -\frac{1}{5}\left(\overbrace{\frac{5}{n\pi}x\sin\frac{n\pi x}{5}\Big|_{-5}^{5}}^{=0} - \frac{5}{n\pi}\int_{-5}^{5}\sin\frac{n\pi x}{5}\,dx = \frac{1}{n\pi}\left(-\frac{5}{n\pi}\right)\cos\frac{n\pi x}{5}\Big|_{-5}^{5} = 0\right.$$

$$b_n = \frac{1}{5}\int_{-5}^{5}(-x)\sin\frac{n\pi x}{5}\,dx = \left| \begin{matrix} u = x & dv = \sin\frac{n\pi x}{5}\,dx \\ du = dx & v = \frac{5}{n\pi}\left(-\cos\frac{n\pi x}{5}\right) \end{matrix} \right| =$$

$$= -\frac{1}{5}\left(\frac{-5}{n\pi}\,x\cos\frac{n\pi x}{5}\Big|_{-5}^{5} + \frac{5}{n\pi}\int_{-5}^{5}\cos\frac{n\pi x}{5}\,dx\right) =$$

$$= \frac{1}{n\pi}\left(5\cos n\pi - (-5)\cos n\pi\right) - \frac{1}{n\pi}\cdot\frac{5}{n\pi}\underbrace{\sin\frac{n\pi x}{5}\Big|_{-5}^{5}}_{=0} =$$

$$= \frac{10}{n\pi}\cos n\pi = \frac{10}{n\pi}(-1)^n$$

Prema tome

$$-x \sim \sum_{n=1}^{\infty}\frac{10}{n\pi}(-1)^n\sin\frac{n\pi x}{5} \quad \text{za } \forall x\in[-5,5]$$

tj. $$-x = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{5} \quad \text{za } \forall x\in[-5,5]$$

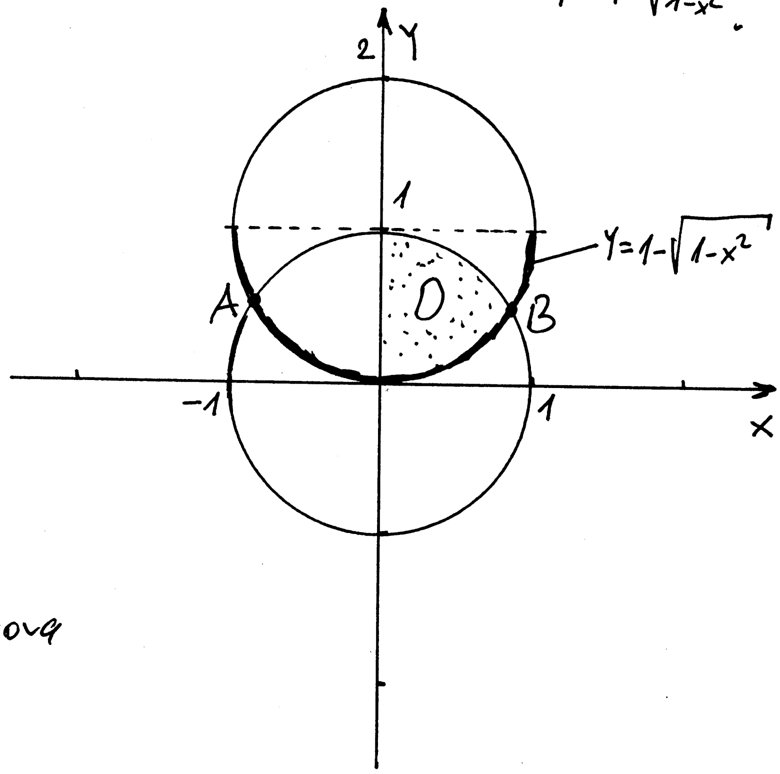
Ako za x uzmemo $x = \frac{1}{10}$ imamo:

$$-\frac{1}{10} = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi\cdot\frac{1}{10}}{5}$$

tj. $$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi}{50} = -\frac{\pi}{100} \quad \text{tražena suma}$$

(#) Izračunati integral $I=\int_0^{\sqrt{3}/2} dx \int_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy$.

Rj: Pokušajmo prvo skicirati oblast integracije D. Primjetimo da se u drugom integralu pojavljuju f-je $y=\sqrt{1-x^2}$ i $y=1-\sqrt{1-x^2}$. Nacrtajmo ih.



$y=\sqrt{1-x^2}$

$y^2=1-x^2$

$x^2+y^2=1$

krug sa centrom u $C(0,0)$ poluprečnika $r=1$

$y=1-\sqrt{1-x^2}$

$y-1=-\sqrt{1-x^2}$

$(y-1)^2=1-x^2$

$x^2+(y-1)^2=1$

krug sa centrom u $C(0,1)$ poluprečnika $r=1$

Pronađimo tačke presjeka ovih krugova

$$\frac{\begin{matrix} x^2+y^2=1 \\ x^2+(y-1)^2=1 \end{matrix}}{\begin{matrix} x^2=1-y^2 \\ x^2+(y-1)^2=1 \end{matrix}}$$

$1-y^2+(y-1)^2=1$

$1-\cancel{y^2}+\cancel{y^2}-2y+\cancel{1}=\cancel{1}$

$1-2y=0$

$2y=1$

$y=\frac{1}{2}$

$x^2+y^2=1$

$x^2=1-y^2$

$x^2=1-\frac{1}{4}$

$x^2=\frac{3}{4}$

$x_{1,2}=\pm\frac{\sqrt{3}}{2}$

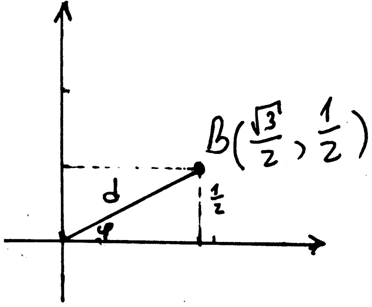
Tačke presjeka su $A(-\frac{\sqrt{3}}{2},\frac{1}{2})$ i $B(\frac{\sqrt{3}}{2},\frac{1}{2})$

Sad možemo konačno nacrtati oblast integracije D.

$$I=\int_0^{\sqrt{3}/2} dx \int_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy = \iint_D \sqrt{x^2+y^2}\, dx\, dy$$

Oblast D ćemo podijeliti na dva dijela D_1 i D_2 pa ćemo imati

$$\iint_D \sqrt{x^2+y^2}\, dx\, dy = \iint_{D_1} \sqrt{x^2+y^2}\, dx\, dy + \iint_{D_2} \sqrt{x^2+y^2}\, dx\, dy.$$

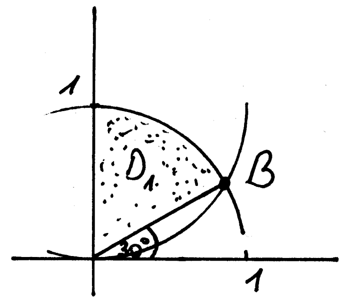


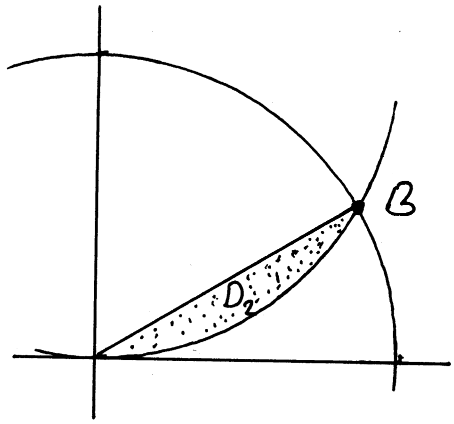
$$d=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$$

$$\sin\varphi=\frac{\frac{1}{2}}{1}$$

$$\sin\varphi=\frac{1}{2}$$

$$\varphi=30^\circ$$

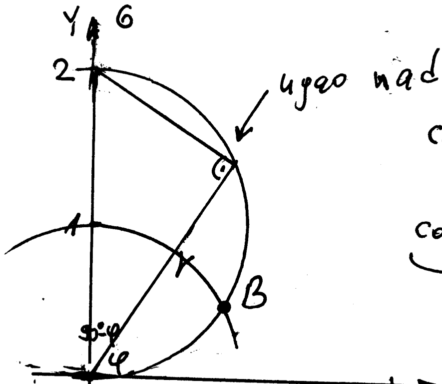




$$\iint\limits_{D_1}\sqrt{x^2+y^2}\,dx\,dy=\left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi\\ y=r\sin\varphi\\ dx\,dy=r\,dr\,d\varphi\end{array}\right.$$

$$\left. D_1 \xrightarrow{\text{transformiše}} D_1':\begin{cases}0\le r\le 1\\ \frac{\pi}{6}\le\varphi\le\frac{\pi}{2}\end{cases}\right| =\iint\limits_{D_1'}\sqrt{r^2}\,r\,dr\,d\varphi=\int_0^1 r^2\,dr\int_{\frac{\pi}{6}}^{\pi/2}d\varphi=$$

$$=\varphi\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cdot\frac{1}{3}r^3\Big|_0^1=\frac{1}{3}\left(\frac{3\pi}{6}-\frac{\pi}{6}\right)=\frac{1}{3}\cdot\frac{2\pi}{6}=\frac{1}{3}\cdot\frac{\pi}{3}=\frac{\pi}{9}$$



ugao nad prečnikom

$$\cos(90^\circ-\varphi)=\frac{r}{2}$$

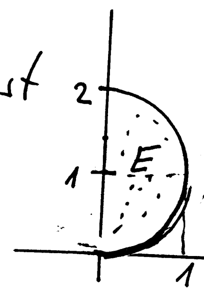
$$\underbrace{\cos 90^\circ\cos\varphi+\sin 90^\circ\sin\varphi}_{\cos(90^\circ-\varphi)}=\frac{r}{2}$$

$$\sin\varphi=\frac{r}{2}$$

$$r=2\sin\varphi$$

Prema tome pomoćna oblast E ima granice

$$E:\begin{cases}0\le r\le 2\sin\varphi\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}$$



Odavde možemo vidjeti polarne granice za D_2

$$\iint\limits_{D_2}\sqrt{x^2+y^2}\,dx\,dy=\left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi\\ y=r\sin\varphi\\ dx\,dy=r\,dr\,d\varphi\end{array}\quad D_2\xrightarrow{\text{transformiše}}D_2':\begin{cases}0\le r\le 2\sin\varphi\\ 0\le\varphi\le\frac{\pi}{6}\end{cases}\right|$$

$$=\iint\limits_{D_2'}r^2\,dr\,d\varphi=\int_0^{\frac{\pi}{6}}d\varphi\int_0^{2\sin\varphi}r^2\,dr=\int_0^{\frac{\pi}{6}}\frac{1}{3}r^3\Big|_0^{2\sin\varphi}d\varphi=\frac{8}{3}\int_0^{\frac{\pi}{6}}\sin^3\varphi\,d\varphi=\overset{\text{za vježbu}}{\dots}=-\sqrt{3}+\frac{16}{9}$$

Prema tome $I=\frac{\pi}{9}+\frac{16}{9}-\sqrt{3}=\frac{\pi+16}{9}-\sqrt{3}$ traženo rješenje

(#) Izračunati pomoću Greenove formule krivolinijski integral

$$I = \oint_C \left(x^2y + \frac{1}{3}y^3 + ye^{xy}\right)dx + \left(x + xe^{xy}\right)dy$$

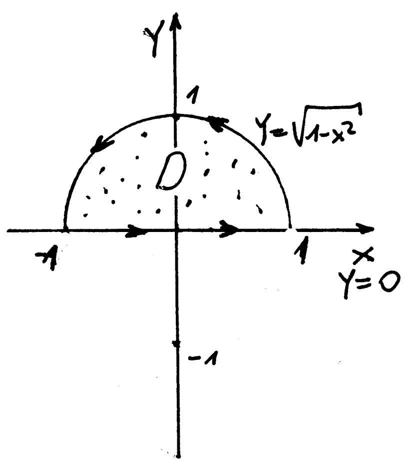
ako je c pozitivno orjentisana kontura određena linijama $y=\sqrt{1-x^2}$, $y=0$.

Rj. Greenova formula glasi $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

gdje je c-zatvorena kontura, S-oblast ograničena konturom

$y=\sqrt{1-x^2}$ $\quad y^2 = 1-x^2$

$x^2+y^2=1$



$$P = x^2y + \frac{1}{3}y^3 + ye^{xy}$$

$$\frac{\partial P}{\partial y} = x^2 + y^2 + e^{xy} + ye^{xy}\cdot x$$

$$Q = x + xe^{xy}$$

$$\frac{\partial Q}{\partial x} = 1 + e^{xy} + xe^{xy}\cdot y$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - x^2 - y^2$$

$$\oint_C \left(x^2y + \frac{1}{3}y^3 + ye^{xy}\right)dx + \left(x + xe^{xy}\right)dy = \left|\begin{array}{l}\text{primjena}\\ \text{Greenove}\\ \text{formule}\end{array}\right| = \iint_D (1-x^2-y^2)\,dx\,dy$$

$$= \left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x = r\cos\varphi\\ y = r\sin\varphi\\ dx\,dy = r\,dr\,d\varphi\end{array} \quad D \xrightarrow{\text{transformise}} D': \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \pi \end{cases}\right| = \iint_{D'} (1-r^2)\,r\,dr\,d\varphi$$

$x^2+y^2=r^2$

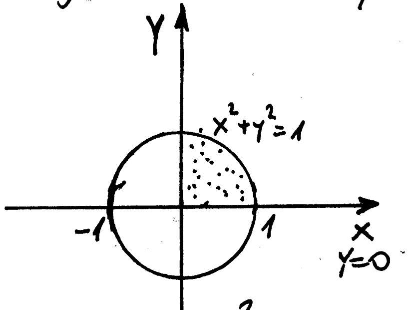
$$= \int_0^1 (r-r^3)\,dr \int_0^\pi d\varphi = \varphi\Big|_0^\pi \int_0^1 (r-r^3)\,dr = \pi\left(\frac{1}{2}r^2\Big|_0^1 - \frac{1}{4}r^4\Big|_0^1\right) = \frac{\pi}{4}$$

traženo rješenje

Izračunati površinski integral $\iint_S xz\,dydz + xy\,dzdx + yz\,dxdy$,

ako je S vanjska strana ~~(omotača)~~ tijela koje pripada prvom oktantu i ograničeno je cilindrom $x^2+y^2=1$, te ravnima $x=0$, $y=0$, $z=0$, $z=2$.

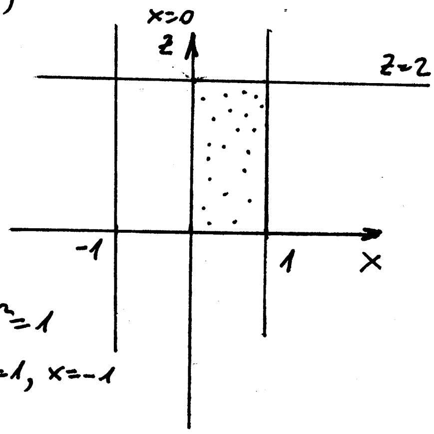
Rj. Skicirajmo dato tijelo. Odredimo prvo presjeke datog tijela sa xOy-ravni, sa xOz-ravni i sa yOz-ravni:



$x^2=1-y^2$

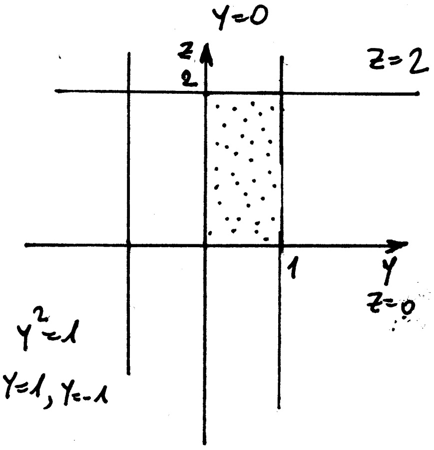
$x=\pm\sqrt{1-y^2}$

$y=\pm\sqrt{1-x^2}$



$x^2=1$

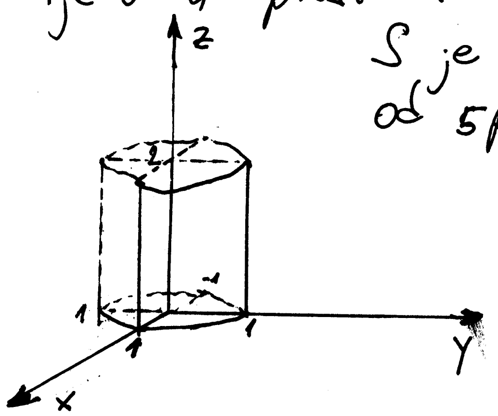
$x=1,\ x=-1$



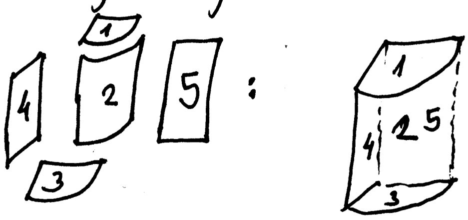
$y^2=1$

$y=1,\ y=-1$

Tijelo u prostoru



S je vanjska strana tijela tj. S se sastoji od 5pet površina



Kako je S zatvorena površina to možemo upotrebiti formulu Gauss-Ostrogradski:

$$\iint_S P\,dydz + Q\,dxdz + R\,dxdy = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz$$

$P(x,y,z)=xz$, $\frac{\partial P}{\partial x}=z$, $Q(x,y,z)=xy$, $\frac{\partial Q}{\partial y}=x$, $R(x,y,z)=yz$, $\frac{\partial R}{\partial z}=y$

$$\iint_S xz\,dydz + xy\,dzdx + yz\,dxdy = \left|\begin{array}{l}\text{Formula}\\ \text{Gauss-}\\ \text{Ostrogradski}\end{array}\right| = \iiint_\Omega (x+y+z)\,dxdydz =$$

$$= \left|\begin{array}{l}\text{uvedimo cilindrične koordinate}\\ x=r\cos\varphi \\ y=r\sin\varphi \qquad \Omega \xrightarrow{\text{transformiše}} \Omega' : \begin{cases} 0\le r\le 1 \\ 0\le\varphi\le\frac{\pi}{2} \\ 0\le z\le 2\end{cases} \\ z=z \\ dxdydz = r\,dr\,d\varphi\,dz\end{array}\right| = \int_0^2 dz \int_0^{\pi/2} d\varphi \int_0^1 (r\cos\varphi + r\sin\varphi + z)\,r\,dr =$$

$$= \int_0^2 dz \int_0^{\pi/2} d\varphi \int_0^1 (r^2\cos\varphi + r^2\sin\varphi + rz)\, dr =$$

$$= \int_0^2 dz \int_0^{\pi/2} \left(\frac{1}{3} r^3 \Big|_0^1 \cos\varphi + \frac{1}{3} r^3 \Big|_0^1 \sin\varphi + \frac{1}{2} r^2 \Big|_0^1 z \right) d\varphi =$$

$$= \int_0^2 dz \int_0^{\pi/2} \left(\frac{1}{3} \cos\varphi + \frac{1}{3} \sin\varphi + \frac{1}{2} z \right) d\varphi =$$

$$= \int_0^2 \left(\frac{1}{3} \sin\varphi \Big|_0^{\frac{\pi}{2}} - \frac{1}{3} \cos\varphi \Big|_0^{\pi/2} + \frac{1}{2} z \varphi \Big|_0^{\frac{\pi}{2}} \right) dz$$

$$= \int_0^2 \left(\frac{1}{3} + \frac{1}{3} + \frac{\pi}{4} z \right) dz = \int_0^2 \left(\frac{2}{3} + \frac{\pi}{4} z \right) dz =$$

$$= \frac{2}{3} z \Big|_0^2 + \frac{\pi}{4} \cdot \frac{1}{2} z^2 \Big|_0^2 = \frac{4}{3} + \frac{\pi}{2}$$

traženo rješenje